Skew-Hermitian matrix. A square matrix A is calledSkew- Hermitian matrix if the transpose of the conjugate of A is equal to negative of A i.e. where $(\overline{A})^T = -A$

Show that, $A = \begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ is Skew-Hermitian. **Solution:** We have to show that, $(\overline{A})^T = -A$ So $\overline{A} = \begin{bmatrix} -3i & 2-i \\ -2-i & i \end{bmatrix}$ then $(\overline{A})^T = \begin{bmatrix} -3i & -2-i \\ 2-i & i \end{bmatrix}$ $\Rightarrow (\overline{A})^T = -\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ $\Rightarrow (\overline{A})^T = -A$	Show that, $A = \begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & -i \\ -2 & -i & 0 \end{bmatrix}$ is Skew-Hermitian. **Solution:** We have to show that, $(\overline{A})^T = -A$ So $\overline{A} = \begin{bmatrix} -i & 1+i & 2 \\ -1+i & -3i & i \\ -2 & i & 0 \end{bmatrix}$ then $(\overline{A})^T = \begin{bmatrix} -i & -1+i & -2 \\ 1+i & -3i & i \\ 2 & i & 0 \end{bmatrix}$ $\Rightarrow (\overline{A})^T = -\begin{bmatrix} i & 1-i & 2 \\ -1-i & 3i & -i \\ -2 & -i & 0 \end{bmatrix}$ $\Rightarrow (\overline{A})^T = -A$

Exercises:

(i) Show that, $B = \begin{bmatrix} i & 1+i & 2-3i \\ -1+i & 2i & 1 \\ -2-3i & -1 & 0 \end{bmatrix}$ is skew- Hermitian and $\overline{B}$ is skew-Hermitian and iB is Hermitian.

i. $A = \begin{bmatrix} 3i & 3+4i & 4-5i \\ -3+4i & -4i & 5+6i \\ -4-5i & -5+6i & 0 \end{bmatrix}$

ii. $A = \begin{bmatrix} 4i & 2-i & 3 \\ -2+i & 0 & 4 \\ -3 & -4 & -3i \end{bmatrix}$

Orthogonal matrix: A is an orthogonal matrix if $AA^T = I$.

Example : $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$ then

$$AA^T = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}\begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 + \sin^2 x & -\cos x \sin x + \sin x \cos x \\ -\sin x \cos x + \cos x \sin x & \sin^2 x + \cos^2 x \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Determine if the following matrix is orthogonal or not.

i. $A = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

ii. $A = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

iii. $A = \frac{1}{7}\begin{bmatrix} 3 & 2 & 6 \\ -6 & 3 & 2 \\ 2 & 6 & -3 \end{bmatrix}$

iv. $A = \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$

v. $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

vi. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

vii. $A = \begin{bmatrix} 0 & 2m & n \\ 1 & m & -n \\ 1 & -m & n \end{bmatrix}$ where $l = \frac{1}{\sqrt{2}}$ $m = \frac{1}{\sqrt{6}}$ and $n = \frac{1}{\sqrt{3}}$

viii. $A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$

ix. $A = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

Involutory Matrix : A matrix is involutory if and only if $A^2 = I$

Examples:

INVOLUTORY MATRIX

A square matrix A is said to be involutory matrix if $A^2 = I$.

For example, if $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I.$$

Hence, A is involutory.

Exercise:

i. $A = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$

ii. $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

iii. $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$

(iii) $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$